Derivative and Integral of the Heaviside Step Function

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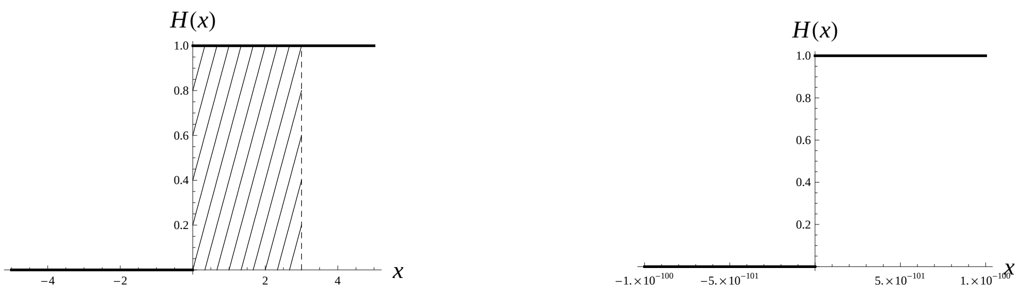
The Setup



(a) Large horizontal scale (b) "Zoomed in"

Figure 1: The Heaviside step function. Note how it doesn't matter how close we get to $x = 0$ the function looks exactly the same.

The Heaviside step function $H(x)$, sometimes called the Heaviside theta function, appears in many places in physics, see [1] for a brief discussion. Simply put, it is a function whose value is zero for $x < 0$ and one for $x > 0$. Explicitly,

$$H(x) = \begin{cases} 0 & x < 0, \\ 1 & x > 0 \end{cases}. \tag{1}$$

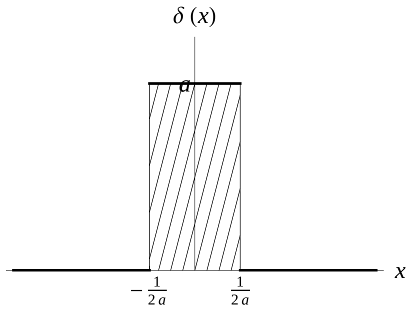
We won't worry about precisely what its value is at zero for now, since it won't effect our discussion, see [2] for a lengthier discussion. Fig. 1 plots $H(x)$. The key point is that crossing zero flips the function from 0 to 1.

Derivative – The Dirac Delta Function

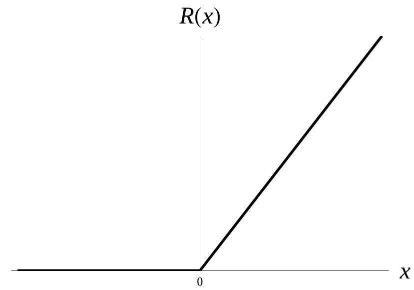
Say we wanted to take the derivative of H. Recall that a derivative is the slope of the curve at at point. One way of formulating this is

$$\frac{dH}{dx} = \lim_{\Delta x \to 0} \frac{\Delta H}{\Delta x}. \tag{2}$$

Now, for any points $x < 0$ or $x > 0$, graphically, the derivative is very clear: H is a flat line in those regions, and the slope of a flat line is zero. In terms of (2), H does not change, so $\Delta H = 0$ and $dH/dx = 0$. But if we pick two points, equally spaced on opposite sides of $x = 0$, say $x_- = -a/2$



(a) Dirac delta function



(b) Ramp function

Figure 2: The derivative (a), and integral (b) of the Heaviside step function.

and $x_+ = a/2$, then $\Delta H = 1$ and $\Delta x = a$. It doesn't matter how small we make a, ΔH stays the same. Thus, the fraction in (2) is

$$\begin{aligned} \frac{dH}{dx} &= \lim_{a \to 0} \frac{1}{a} \\ &= \infty. \end{aligned} \tag{3}$$

Graphically, again, this is very clear: H *jumps* from 0 to 1 at zero, so it's slope is essentially vertical, i.e. infinite. So basically, we have

$$\delta(x) \equiv \frac{dH}{dx} = \begin{cases} 0 & x < 0 \\ \infty & x = 0 \\ 0 & x > 0 \end{cases}. \tag{4}$$

This function is, loosely speaking, a "Dirac Delta" function, usually written as $\delta(x)$, which has seemingly endless uses in physics.

We'll note a few properties of the delta function that we can derive from (4). First, integrating it from $-\infty$ to any $x_- < 0$:

$$\begin{aligned} \int_{-\infty}^{x_-} \delta(x) dx &= \int_{-\infty}^{x_-} \left(\frac{dH}{dx} \right) dx \\ &= H(x_-) - H(-\infty) \\ &= 0 \end{aligned} \tag{5}$$

since $H(x_-) = H(-\infty) = 0$. On the other hand, integrating the delta function to any point greater than $x = 0$:

$$\begin{aligned} \int_{-\infty}^{x_+} \delta(x) dx &= \int_{-\infty}^{x_+} \left(\frac{dH}{dx} \right) dx \\ &= H(x_+) - H(-\infty) \\ &= 1 \end{aligned} \tag{6}$$

since $H(x_+) = 1$.

At this point, I should point out that although the delta function blows up to infinity at $x = 0$, it still has a finite integral. An easy way of seeing how this is possible is shown in Fig. 2(a). If the width of the box is $1/a$ and the height is a, the area of the box (i.e. its integral) is 1, no matter how large a is. By letting a go to infinity we have a box with infinite height, yet, when integrated, has finite area.

Integral – The Ramp Function

Now that we know about the derivative, it's time to evaluate the integral. I have two methods of doing this. The most straightforward way, which I first saw from Prof. T.H. Boyer, is to integrate H piece by piece. The integral of a function is the area under the curve,[1] and when $x < 0$ there is no area, so the integral from $-\infty$ to any point less than zero is zero. On the right side, the integral to a point x is the area of a rectangle of height 1 and length x, see Fig. 1(a). So, we have

$$\int_{-\infty}^{x} H\,dx = \begin{cases} 0 & x < 0, \\ x & x > 0 \end{cases}. \tag{7}$$

We'll call this function a "ramp function," $R(x)$. We can actually make use of the definition of H and simplify the notation:

$$R(x) \equiv \int H\,dx = xH(x) \tag{8}$$

since $0 \times x = 0$ and $1 \times x = x$. See Fig. 2(b) for a graph – and the reason for calling this a "ramp" function.

But I have another way of doing this which makes use of a trick that's often used by physicists: **We can always add zero for free**, since *anything* $+ 0 =$ *anything*. Often we do this by adding and subtracting the same thing,

$$A = (A + B) - B, \tag{9}$$

for example. But we can use the delta function (4) to add zero in the form

$$0 = x\,\delta(x). \tag{10}$$

Since $\delta(x)$ is zero for $x \neq 0$, the x part doesn't do anything in those regions and this expression is zero. And, although $\delta(x) = \infty$ at $x = 0$, $x = 0$ at $x = 0$, so the expression is still zero.

So we'll add this on to H:

$$\begin{aligned} H &= H + 0 \\ &= H + x\,\delta(x) \\ &= H + x\frac{dH}{dx} \qquad \text{by (4)} \\ &= \frac{dx}{dx}H + x\frac{dH}{dx} \\ &= \frac{d}{dx}\left[xH(x)\right], \end{aligned} \tag{11}$$

where the last step follows from the "product rule" for differentiation. At this point, to take the integral of a full differential is trivial, and we get (8).

[1] To be completely precise, it's the (signed) area between the curve and the line $x = 0$.

References

[1] M. Springer. Sunday function [online]. February 2009. Available from: http://scienceblogs.com/builtonfacts/2009/02/sunday_function_22.php [cited 30 June 2009].

[2] E.W. Weisstein. Heaviside step function [online]. Available from: http://mathworld.wolfram.com/HeavisideStepFunction.html [cited 30 June 2009].